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A new type of Hopf algebra which is neither commutative nor cocommutative

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Abstract. In this paper a new determinant-cofactor method is used to impose the crucial constraints on the entries of the multiparametric R-matrices mentioned by Yu I Manin. We obtain the quotient Hopf algebras from the YBZF algebras which are defined by the restricted R-matrices. A subclass of algebra with the q-parameter is also discussed.

J Fröhlich discussed the dual algebra relations in [2]. In the present paper, we define and investigate a new type of Hopf algebra, which generalizes the dual algebra relations to the multiparametric deformations of the general linear groups.

Although the R-matrices used in this paper were mentioned by Yu I Manin in [1], we obtain the crucial constraints on the entries of the R-matrices in order to construct the new type of quantum groups. Yu I Manin gave the abstract definition of the quantum determinant in [1]. However, we introduce the interesting determinant-cofactor method to obtain the explicit forms of the quantum determinant and antipodal map in our cases.

A special subclass of algebra with the q-parameter, discussed in this paper, provides a new member of compact matrix pseudogroups, which were proposed by Woronowicz in [3, 4].

First, we briefly review some basic facts of the Yang–Baxter–Zamolochikov–Faddeev (YBZF) algebras of R-matrices (cf [5–7]):

(i) The R-matrix. Let K be a field. A matrix $R \in \mathrm{gl}(n^2, K)$, for some $n \in N$, is called a R-matrix if R satisfies the Yang–Baxter equation,

$$R^{(12)}R^{(23)}R^{(12)} = R^{(23)}R^{(12)}R^{(23)}. \tag{1}$$

(ii) The YBZF algebra of an R-matrix. Let $R \in \mathrm{gl}(n^2, K)$ be an R-matrix. The YBZF algebra $\mathscr{A}_R$ of R is defined as

$$\mathscr{A}_R \triangleq K\langle T_{ij} | i, j = 1, 2, \ldots, n\rangle / K\langle R \cdot T \otimes T - T \otimes T \cdot R\rangle. \tag{2}$$

(iii) The YBZF algebra of an R-matrix is a bialgebra.

(a) The coproduct $\Delta : \mathscr{A}_R \to \mathscr{A}_R \otimes \mathscr{A}_R$ is a homomorphism which satisfies

$$\Delta(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj} \qquad \Delta(e) = e \otimes e. \tag{3}$$

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(*b*) The co-unit $\varepsilon : \mathscr{A}_R \to K$ is a homomorphism which satisfies

$$\varepsilon(T_{ij}) = \delta_{ij} \qquad \varepsilon(e) = 1. \tag{4}$$

It is well known that the quotient algebras of the YBZF algebras of the R-matrices discussed by Manin, Drinfeld and others (cf [1, 6]) are Hopf algebras which are called quantum groups.

In this paper, we investigate another type of R-matrix, which was mentioned by Manin in [1]. We then introduce the interesting determinant–cofactor method to obtain the crucial constraints on the entries of the R-matrices in order to construct the new quantum groups from the YBZF algebras.

Proposition 1. Let $A \in \mathrm{gl}(n, K)$ and suppose that $R_A \in \mathrm{gl}(n^2, K)$ satisfies

$$(R_A)_{ij,kl} = a_{ij}\delta_{il}\delta_{jk} \qquad \forall i, j, k, l = 1, 2, \ldots, n. \tag{5}$$

Then R_A is a R-matrix.

Proof. One can prove proposition 1 after a direct calculation. □

Corollary 2. If $A \in \mathrm{gl}(n, K)$ and for all i, j, $a_{ij} \neq 0$, where a_{ij} is the (i, j) entry of the matrix A, then the generators $\{T_{ij} | i, j = 1, 2, \ldots, n\}$ of the YBZF algebra $\mathscr{A}_{R_A}$ of R_A satisfy at least the following relations:

$$T_{ij}T_{kl} = \frac{a_{lj}}{a_{ki}} T_{kl}T_{ij} \qquad \forall i, j, k, l = 1, 2, \ldots, n. \tag{6}$$

Proof. By the definition of R_A and $\mathscr{A}_{R_A}$, the relations (6) follow from a direct calculation. □

Remark. If one suppose that $T_{ij}T_{kl} \neq 0$, for all i, j, k, l, then one must have $a_{ij}a_{ji} =$ constant $\neq 0$, for all i, j. For convenience, we now let $a_{ij}a_{ji} = 1$, for all i, j. In particular, if the field K is the complex field C, then we have $a_{ii} = 1$ or $a_{ii} = -1$. The exact choice of a_{ii} will be determined in theorem 4 and the remark thereafter.

In order to construct the quotient Hopf algebra of $\mathscr{A}_R$, we must define the antipodal map $S : \mathscr{A}_R \to \mathscr{A}_R$, which is an antihomomorphism satisfying

$$T \cdot S(T) = S(T) \cdot T = eI_{n\times n}. \tag{7}$$

We see that the definition of the antipode S is to get the inverse matrix T^{-1} of the $\mathscr{A}_R$-valued matrix $T = (T_{ij})_{n\times n}$. As we know, the standard method to obtain the inverse matrix B^{-1} of the number-valued matrix B is to calculate the adjoint matrix of B in terms of the determinant and the algebraic cofactors of B. Hence, it is natural for us to generalize the concepts and methods of determinant and algebraic cofactors in our cases.

Definition 3. (The row determinant and the column determinant of the $\mathscr{A}_{R_A}$-valued matrices.) Suppose that $A \in \mathrm{gl}(n, K)$ satisfies

$$a_{ij}a_{ji} = 1 \qquad a_{ii} = 1 \qquad \forall i, j = 1, 2, \ldots, n. \tag{8}$$

Let $\mathscr{A}_{R_A}$ be the YBZF algebra of R_A and $\{T_{ij} | i, j = 1, 2, \ldots, n\}$ be the generators of $\mathscr{A}_{R_A}$. Now we define the two sets of the $\mathscr{A}_{R_A}$-valued matrices as

$$\begin{aligned} \mathscr{A}_{R_A}^{(r)} &\triangleq \{B \in \mathrm{gl}(n, \mathscr{A}_{R_A}) | \text{each row of } B \text{ is } (T_{i1}, T_{i2}, \ldots, T_{in}) \text{ for some } i\} \\ \mathscr{A}_{R_A}^{(c)} &\triangleq \{B \in \mathrm{gl}(n, \mathscr{A}_{R_A}) | \text{each column of } B \text{ is } (T_{1j}, T_{2j}, \ldots, T_{nj}) \text{ for some } j\} \end{aligned} \tag{9}$$

and let

$$\tilde{a}_{ij} \triangleq \begin{cases} 1 & \text{if } i < j \\ a_{ij} & \text{if } i \geq j. \end{cases} \tag{10}$$

Moreover, we define the row determinant $\det^r$ and the column determinant $\det^c$ of $B \in \mathscr{A}_{R_A}^{(r)}$ and $B' \in \mathscr{A}_{R_A}^{(c)}$, respectively, as follows:

(i) For $B \in \mathscr{A}_{R_A}^{(r)}$, we define the row determinant of B as

$$\det{}^r B \triangleq \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_r(\sigma) B_{1\sigma(1)} \ldots B_{n\sigma(n)} \tag{11}$$

where $B = (B_{ij})_{n\times n}$, S_n is the nth permutation group and $a_r(\sigma)$ is defined as

$$a_r(\sigma) \triangleq \prod_{i<j} \tilde{a}_{\sigma(i)\sigma(j)}. \tag{12}$$

(ii) For $B' \in \mathscr{A}_{R_A}^{(c)}$, we define the column determinant of B' as

$$\det{}^c B' \triangleq \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_c(\sigma) B'_{\sigma(1)1} \ldots B'_{\sigma(n)n} \tag{13}$$

where $B' = (B'_{ij})_{n\times n}$ and $a_c(\sigma)$ is defined as the inverse of $a_r(\sigma)$, i.e.

$$a_c(\sigma) \triangleq a_r(\sigma)^{-1}. \tag{14}$$

Remark. For all $\sigma \in S_n$, the set $\{(i, j) | i < j, \sigma(i) > \sigma(j)\}$ is the set of the reversed-order pairs of σ. By the definition of $a_r(\sigma)$ and $a_c(\sigma)$, we see that the row determinant and the column determinant express the reversed-order action of the permutation $\sigma \in S_n$.

Theorem 4. (The properties of the row determinant and the column determinant.) Suppose that $A \in \mathrm{gl}(n, K)$ satisfies (8). Then

(i) $$\tilde{a}_{ij} a_{ji} = \tilde{a}_{ji} \qquad \forall i, j = 1, 2, \ldots, n. \tag{15}$$

(ii) If for $B \in \mathscr{A}_{R_A}^{(r)}$, the kth row and the $(k+1)$th row of B are $(T_{i1}, \ldots, T_{in})$ and $(T_{j1}, \ldots, T_{jn})$ for some i, j respectively, and $\tilde{B}$ denotes the matrix given by exchanging the kth row and the $(k+1)$th row of B, then

$$\det{}^r \tilde{B} = -a_{ji} \det{}^r B. \tag{16}$$

In particular, if $i = j$, i.e. B has equal neighbouring rows, then $\det^r B = 0$. Moreover, if B has equal rows, then $\det^r B = 0$.

(iii) If for $B' \in \mathscr{A}_{R_A}^{(c)}$, the kth column and the $(k+1)$th column of B' are $(T_{1i}, \ldots, T_{ni})$ and $(T_{1j}, \ldots, T_{nj})$ for some i, j respectively, and $\tilde{B}'$ denotes the matrix given by exchanging the kth column and the $(k+1)$th column of B', then

$$\det{}^c \tilde{B}' = -a_{ij} \det{}^c B'. \tag{17}$$

In particular, if $i = j$, i.e. B' has equal neighbouring columns, then $\det^c B' = 0$. Moreover, if B' has equal columns, then $\det^c B' = 0$.

(iv) $T \in \mathscr{A}_{R_A}^{(r)} \cap \mathscr{A}_{R_A}^{(c)}$ and

$$\det{}^r T = \det{}^c T \tag{18}$$

where $T = (T_{ij})_{n\times n}$ is the generator matrix of $\mathscr{A}_{R_A}$.

Proof.

(i) It follows from (10) that $\tilde{a}_{ij}a_{ji} = \tilde{a}_{ji}$ for all i, j.

(ii) If σ_k denotes the pair permutation $(k, k+1)$ of S_n, then for all $\sigma' \in S_n$, we get

$$a_r(\sigma' \circ \sigma_k) = \prod_{i<j} \tilde{a}_{\sigma'\circ\sigma_k(i)\sigma'\circ\sigma_k(j)}$$

$$= \prod_{\substack{i<j,\\ i,j\neq k,k+1}} \tilde{a}_{\sigma'(i)\sigma'(j)} \prod_{\substack{i=k,\\ j>k+1}} \tilde{a}_{\sigma'(k+1)\sigma'(j)} \prod_{\substack{i=k+1,\\ j>k+1}} \tilde{a}_{\sigma'(k)\sigma'(j)}$$

$$\times \prod_{\substack{i<k,\\ j=k+1}} \tilde{a}_{\sigma'(i)\sigma'(k)} \prod_{\substack{i<k,\\ j=k}} \tilde{a}_{\sigma'(i)\sigma'(k+1)}\tilde{a}_{\sigma'(k+1)\sigma'(k)}$$

$$= a_r(\sigma') \frac{\tilde{a}_{\sigma'(k+1)\sigma'(k)}}{\tilde{a}_{\sigma'(k)\sigma'(k+1)}}$$

$$= a_r(\sigma') a_{\sigma'(k+1)\sigma'(k)}$$

by definition (12). Moreover, by the definition of the row determinant, we then obtain

$$\det{}^r \tilde{B} = \sum_{\sigma\in s_n} \operatorname{sgn}(\sigma) a_r(\sigma) B_{1\sigma(1)} \ldots B_{(k+1)\sigma(k)} B_{k\sigma(k+1)} \ldots B_{n\sigma(n)}$$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_r(\sigma) B_{1\sigma(1)} \ldots T_{j\sigma(k)} T_{i\sigma(k+1)} \ldots B_{n\sigma(n)}$$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_r(\sigma) \frac{a_{\sigma(k+1)\sigma(k)}}{\sigma_{ij}} B_{1\sigma(1)} \ldots T_{i\sigma(k+1)} T_{j\sigma(k)} \ldots B_{n\sigma(n)}$$

$$= \sum_{\sigma'\in S_n} \operatorname{sgn}(\sigma'\circ\sigma_k) a_r(\sigma'\circ\sigma_k) \frac{a_{\sigma'(k)\sigma'(k+1)}}{a_{ij}} B_{1\sigma'(1)} \ldots T_{i\sigma'(k)} T_{j\sigma'(k+1)} \ldots B_{n\sigma'(n)}$$

$$= \sum_{\sigma'\in S_n} -\operatorname{sgn}(\sigma') a_r(\sigma') a_{\sigma'(k+1)\sigma'(k)} \frac{a_{\sigma'(k)\sigma'(k+1)}}{a_{ij}}$$

$$\times B_{1\sigma'(1)} \ldots B_{k\sigma'(k)} B_{(k\times 1)\sigma'(k+1)} \ldots B_{n\sigma'(n)}$$

$$= -a_{ji} \det{}^r B.$$

In particular, if $i=j$, then we get $\det^r B = 0$ by $a_{ii}=1$ and $\tilde{B}=B$. Hence, if B has two equal rows, then we prove that $\det^r B = 0$ by identity (16).

(iii) The proof of (17) is similar to that of (16).

(iv) For all $\sigma \in S_n$, we get

$$T_{\sigma(1)1} \ldots T_{\sigma(n)n} = \prod_{k=1}^{n} \left(\prod_{p=1, p\neq\sigma^{-1}(1),\ldots,\sigma^{-1}(k-1)}^{\sigma^{-1}(k)-1} \frac{a_{\sigma^{-1}(k)p}}{a_{k\sigma(p)}} \right) T_{1\sigma^{-1}(1)} \ldots T_{n\sigma^{-1}(n)}$$

$$= \prod_{k=1}^{n} \left(\prod_{p<\sigma^{-1}(k),\sigma(p)>k} \frac{a_{\sigma^{-1}(k)p}}{a_{k\sigma(p)}} \right) T_{1\sigma^{-1}(1)} \ldots T_{n\sigma^{-1}(n)}.$$

Thus we obtain

$$\prod_{k=1}^{n} \left(\prod_{p<\sigma^{-1}(k),\sigma(p)>k} a_{k\sigma(p)} \right) T_{\sigma(1)1} \ldots T_{\sigma(n)n}$$

$$= \prod_{k=1}^{n} \left(\prod_{p<\sigma^{-1}(k),\sigma(p)>k} a_{\sigma^{-1}(k)p} \right) T_{1\sigma^{-1}(1)} \ldots T_{n\sigma^{-1}(n)}$$

i.e.

$$a_c(\sigma)T_{\sigma(1)1}\dots T_{\sigma(n)n}=a_r(\sigma^{-1})T_{1\sigma^{-1}(1)}\dots T_{n\sigma^{-1}(n)}$$

for all $\sigma\in S_n$. This implies at once that $\det^r T=\det^c T$.

Remark. In the proof of (2) in theorem 4, the exact choice of a_{ii} guarantees that $\det^r B$ ($\det^c B'=0$) if B (B') has two equal rows (columns).

Definition 5. (The determinant and the algebraic cofactors of the generator-matrix T of $\mathscr{A}_{R_A}$.) Suppose that $\mathscr{A}\in \mathrm{gl}(n,K)$ satisfies (8). Then

(i) The determinant of T is defined as

$$\det T\triangleq\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)a_r(\sigma)T_{1\sigma(1)}\dots T_{n\sigma(n)}$$
$$\equiv\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)a_c(\sigma)T_{\sigma(1)1}\dots T_{\sigma(n)n}. \tag{19}$$

(ii) Four types of algebraic cofactors of T are defined as follows:

(a) The left-row algebraic cofactor of T_{ij} is defined as

$$A^{r,L}(T_{ij})\triangleq\sum_{\substack{\sigma\in S_n,\\ \sigma(i)=j}}\mathrm{sgn}(\sigma)a_r(\sigma)\left(\prod_{k=1}^{i-1}\frac{a_{j\sigma(k)}}{a_{ik}}\right)T_{1\sigma(1)}\dots\hat{T}_{ij}\dots T_{n\sigma(n)} \tag{20}$$

where $\hat{T}_{ij}$ denotes T_{ij} deleted in (20).

(b) The right-row algebraic cofactor of T_{ij} is defined as

$$A^{r,R}(T_{ij})\triangleq\sum_{\substack{\sigma\in S_n,\\ \sigma(i)=j}}\mathrm{sgn}(\sigma)a_r(\sigma)\left(\prod_{k=i+1}^{n}\frac{a_{\sigma(k)j}}{a_{ki}}\right)T_{1\sigma(1)}\dots\hat{T}_{ij}\dots T_{n\sigma(n)}. \tag{21}$$

(c) The left-column algebraic cofactor of T_{ij} is defined as

$$A^{c,L}(T_{ij})\triangleq\sum_{\substack{\sigma\in S_n,\\ \sigma(j)=i}}\mathrm{sgn}(\sigma)a_c(\sigma)\left(\prod_{k=1}^{j-1}\frac{a_{jk}}{a_{i\sigma(k)}}\right)T_{\sigma(1)1}\dots\hat{T}_{ij}\dots T_{\sigma(n)n}. \tag{22}$$

(d) The right-column algebraic cofactor of T_{ij} is

$$A^{c,R}(T_{ij})\triangleq\sum_{\substack{\sigma\in S_n,\\ \sigma(j)=i}}\mathrm{sgn}(\sigma)a_c(\sigma)\left(\prod_{k=j+1}^{n}\frac{a_{kj}}{a_{\sigma(k)i}}\right)T_{\sigma(1)1}\dots\hat{T}_{ij}\dots T_{\sigma(n)n}. \tag{23}$$

Theorem 6. (The properties of determinant and algebraic cofactors of T.) Suppose that $A\in\mathrm{gl}(n,K)$ satisfies (8). Then, for all i,j:

(i)
$$T_{ij}\det T=\prod_{k=1}^{n}\frac{a_{kj}}{a_{ki}}\det T T_{ij} \tag{24}$$

(ii)
$$A^{r,L}(T_{ij})=\prod_{k=1}^{n}\frac{a_{jk}}{a_{ik}}A^{r,R}(T_{ij})$$
$$A^{c,R}(T_{ij})=\prod_{k=1}^{n}\frac{a_{jk}}{a_{ik}}A^{c,L}(T_{ij}). \tag{25}$$

(iii)
$$\begin{aligned}
&\sum_{k=1}^{n} T_{ik}A^{r,L}(T_{jk}) = \delta_{ij} \det T \\
&\sum_{k=1}^{n} A^{r,R}(T_{ik})T_{jk} = \delta_{ij} \det T \\
&\sum_{k=1}^{n} T_{ki}A^{c,L}(T_{kj}) = \delta_{ij} \det T \\
&\sum_{k=1}^{n} A^{c,R}(T_{ik})T_{kj} = \delta_{ij} \det T.
\end{aligned} \tag{26}$$

Proof. (i) It follows from (6) and the definition of det T that identity (24) holds.

(ii) For all $\sigma \in S_n$, we get

$$\prod_{k=1}^{n} \frac{a_{jk}}{a_{ik}} = \prod_{k=1}^{n} \frac{a_{j\sigma(k)}}{a_{ik}} = \prod_{k=1}^{i-1} \frac{a_{j\sigma(k)}}{a_{ik}} \prod_{k=i+1}^{n} \frac{a_{j\sigma(k)}}{a_{ik}} \frac{a_{j\sigma(i)}}{a_{ii}}.$$

If $\sigma(i) = j$, then we obtain

$$\prod_{k=1}^{i-1} \frac{a_{j\sigma(k)}}{a_{ik}} = \prod_{k=1}^{n} \frac{a_{jk}}{a_{ik}} \prod_{k=i+1}^{n} \frac{a_{\sigma(k)j}}{a_{ki}}.$$

With the definition of algebraic cofactors, this implies (25) immediately.

(iii) By (ii) and (iii) in theorem 4, we get (26) after a direct calculation.

Corollary 7. Suppose that $A \in \mathrm{gl}(n, K)$ satisfies (8) and

$$\prod_{k=1}^{n} \frac{a_{ki}}{a_{kj}} = 1 \qquad \forall i, j = 1, 2, \ldots, n. \tag{27}$$

Then:

(i)
$$T_{ij} \det T = \det T\, T_{ij} \qquad \forall i, j = 1, 2, \ldots, n. \tag{28}$$

(ii)
$$A^{r,L}(T_{ij}) = A^{r,R}(T_{ij}) \qquad A^{c,R}(T_{ij}) = A^{c,L}(T_{ij}). \tag{29}$$

Theorem 8. Suppose that $A \in \mathrm{gl}(n, K)$ satisfies

$$a_{ij}a_{ji} = 1 \qquad a_{ii} = 1 \qquad \prod_{k=1}^{n} \frac{a_{ki}}{a_{kj}} = 1 \qquad \forall i, j = 1, 2, \ldots, n. \tag{30}$$

Then:

(i)
$$\begin{aligned}
T \cdot A^{r,L}(T)^t &= A^{c,R}(T)^t \cdot T \\
&= A^{r,R}(T) \cdot T^t = T^t \cdot A^{c,L}(T) \\
&= (\det T) I_{n \times n}
\end{aligned} \tag{31}$$

where t denotes the transpose of the matrix. In particular, if det T has an inverse element in $\mathscr{A}_{R_A}$, then

$$A^{r,L}(T) = A^{r,R}(T) = A^{c,L}(T) = A^{c,R}(T). \tag{32}$$

(ii)
$$A^{r,L}(T_{ij})a^{r,L}(T_{kl}) = \frac{a_{lj}}{a_{ki}} A^{r,L}(T_{kl})A^{r,L}(T_{ij}) \qquad \forall i, j, k, l = 1, 2, \ldots, n. \tag{33}$$

(iii) Let $\mathscr{A} \triangleq \mathscr{A}_{R_A}/K\langle \det T - e\rangle$ be the quotient algebra of the YBZF algebra $\mathscr{A}_{R_A}$. Then $\mathscr{A}$ is a Hopf algebra whose antipodal map $S:\mathscr{A}\to\mathscr{A}$ is an antihomomorphism which satisfies

$$S(e)=e \qquad S(T_{ij})=A^{r,L}(T_{ji}). \tag{34}$$

In particular, $S^2=\mathrm{id}$.

Proof. (i) it follows from (iii) in theorem 6(iii) that identity (31) holds. By (28) and det T's being inverse in $\mathscr{A}_{R_A}$, we get (32).

(ii) Since for all i, j, p, q,

$$T_{1\sigma(1)}\ldots\hat{T}_{ij}\ldots T_{n\sigma(n)}T_{pq}=\prod_{r=1}^{n}\frac{a_{q\sigma(r)}}{a_{pr}}\frac{a_{pi}}{a_{qj}}T_{pq}T_{1\sigma(1)}\ldots\hat{T}_{ij}\ldots T_{n\sigma(n)}$$

we get

$$T_{1\sigma(1)}\ldots\hat{T}_{ij}\ldots T_{n\sigma(n)}T_{1\sigma'(1)}\ldots\hat{T}_{kl}\ldots T_{n\sigma'(n)}$$
$$=\frac{a_{lj}}{a_{ki}}T_{1\sigma'(1)}\ldots\hat{T}_{kl}\ldots T_{n\sigma'(n)}T_{1\sigma(1)}\ldots\hat{T}_{ij}\ldots T_{n\sigma(n)}$$

for all $\sigma, \sigma'\in S_n$. By the definition of the left-row algebraic cofactor of T_{ij}, we then obtain (33).

(iii) Identities (26), (28), (32) and (33) imply that the quotient algebra $\mathscr{A}$ is a bialgebra and the extension of antihomomorphism S in (34) is the antipodal map of $\mathscr{A}$. Hence $\mathscr{A}$ is a Hopf algebra whose antipode is S. In particular, we get $S^2=\mathrm{id}$ by (34) and (31).

Remark. (i) The quantum space $A^{n|0}$ and the Frobenius space $A^{0|n}$ (the dual of the quantum space $A^{n|0}$ of the quantum group $\mathscr{A}$) are defined as:

$$A^{n|0}=K\langle v_1, v_2, \ldots, v_n\rangle/K\langle v_iv_j-a_{ji}v_jv_i\rangle$$

and

$$A^{0|n}=K\langle \xi_1, \xi_2, \ldots, \xi_n\rangle/K\langle \xi_i\xi_j+a_{ij}\xi_j\xi_i, \xi_j^2\rangle$$

respectively. By the abstract definition of quantum determinant in [1], we also obtain (19).

(ii) The comodule τ of $\mathscr{A}_{R_A}$ on $A^{n|0}$ is defined as

$$\tau: \qquad A^{n|0}\to A^{n|0}\otimes\mathscr{A}_{R_A}$$
$$v_i\mapsto\sum_{k=1}^{n}v_k\otimes T_{ki}.$$

It is easy to prove that τ is an algebraic homomorphism.

Proposition 9. (The special subclass with q-parameter.) Let K be a field. For all $q\neq 0\in K$ and for all $n\in N$, suppose that $A(q)\in \mathrm{gl}(2n+1, K)$ satisfies

$$a_{ij}(q)\triangleq q^{\mathrm{sgn}(j-i)(-1)^{j-i+1}} \tag{35}$$

where sgn is the sign function of integers. Then $A(q)$ satisfies (30).

Theorem 10. (A new member of compact matrix pseudogroups.) Let C be the complex field. Suppose that q is a non-zero real number. For all $n \in N$, let $A(q) \in \mathrm{gl}(2n+1, C)$ be defined as (35). Then the quotient Hopf algebra $\mathscr{A}_q$ of the YBZF algebra $\mathscr{A}_{R_{A(q)}}$ of the R-matrix $R_{A(q)}$ is a Hopf-$*$ algebra. In particular:

(i) The map $*: \mathscr{A}_{R_{A(q)}} \to \mathscr{A}_{R_{A(q)}}$ is an anti-involution of $\mathscr{A}_{R_{A(q)}}$

$$*(T_{ij}) \triangleq T_{\omega(i)\omega(j)} \tag{36}$$

where the permutation $\omega \in S_{2n+1}$ is defined as

$$\omega(i) \triangleq 2n+2-\mathrm{i} \qquad \forall \mathrm{i} = 1, 2, \ldots, 2n+1. \tag{37}$$

(ii) The map $*$ is compatible with the Hopf algebras structure of $\mathscr{A}_q$. Moreover, the Hopf algebra $\mathscr{A}_q$ equipped with the map $*$ is a Hopf-$*$ algebra.

Proof. (i) Since we get

$$a_{\omega(i)\omega(j)} = a_{ji} \tag{38}$$

where ω is defined as (37), then we have

$$*(T_{ki})*(T_{ij}) = \frac{a_{ij}}{a_{ki}} *(T_{ij})*(T_{ki}).$$

By $\mathrm{sgn}(\omega \circ \sigma \circ \omega) = \mathrm{sgn}(\sigma)$ and $a_r(\omega \circ \sigma \circ \omega) = a_r(\sigma)$, we then obtain $*(\det T) = \det T$. Moreover, by the definition of the map $*$ on $\{T_{ij} \mid i, j = 1, 2, \ldots, 2n+1\}$ as (36), we extend the map $*$ to an anti-involution of $\mathscr{A}_{R_{A(q)}}$.

(ii) We see from the definition of $\mathscr{A}_{R_{A(q)}}$ in theorem 8 and (38) that the map $*$ is compatible with the Hopf algebra structure of $\mathscr{A}_q$. Thus the Hopf algebra $\mathscr{A}_q$ equipped with the map $*$ is a Hopf-$*$ algebra. □

Remark. Theorem 10 shows that $\mathscr{A}_q$ is a new member in the category of the compact matrix pseudogroups (cf [3, 4]). The concept of the corresponding non-commutative differential geometry is of interest for further investigation.

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References

[1] Manin Yu I 1988 *Quantum Groups and Noncommutative Geometry* (Montreal: Les Publications CRM)
[2] Fröhlich J 1988 Statistics of fields, the Yang–Baxter equations and the theory of knots and links *Preprint* ETH-Hönggerberg C-H 8093, Zurich

[3] Woronowicz S L 1987 Twisted SU(2)-group, an example of noncommutative differential calculus *Publ. RIMS, Kyoto University* **23** 17-8

[4] Woronowicz S L 1987 Compact matrix pseudogroups *Commun. Math. Phys.* **111** 613-65

[5] de Vega H J 1986 YBZF algebra, Kac-Moody algebras and integrable theories, *Topological and Geometrical Methods in Fields Theory, Symposium in Espoo, Finland*

[6] Drinfeld V G 1986 Quantum groups *Proc. Int. Congr. Math.* **1** 798-820

[7] Faddeev L D, Reshetikhin N Y and Takhtajan L A 1987 Quantization of Lie groups and Lie algebras *Preprint* LOMI